

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

J. Math. Anal. Appl. 328 (2007) 821–829

---



---

*Journal of*  
 MATHEMATICAL  
 ANALYSIS AND  
 APPLICATIONS
 

---



---

[www.elsevier.com/locate/jmaa](http://www.elsevier.com/locate/jmaa)

# Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications

Hafiz Fukhar-ud-din<sup>a</sup>, Safeer Hussain Khan<sup>b,\*</sup>

<sup>a</sup> *Department of Mathematics, Islamia University, Bahawalpur, Pakistan*

<sup>b</sup> *Department of Mathematics and Physics, Faculty of Arts and Science, Qatar University, Doha 2713, Qatar*

Received 4 May 2005

Available online 11 July 2006

Submitted by William F. Ames

---

## Abstract

In this paper, we consider a generalized iterative process with errors to approximate the common fixed points of two asymptotically quasi-nonexpansive mappings. A convergence theorem has been obtained which generalizes a known result. This theorem has then been used to prove another convergence theorem which, in turn, generalizes a number of results.

© 2006 Elsevier Inc. All rights reserved.

**Keywords:** Iterates with errors; Asymptotically quasi-nonexpansive mapping; Common fixed point; Condition (A'); Strong convergence

---

## 1. Introduction

Let  $E$  be a real Banach space with  $C$  its nonempty subset. Recall that a mapping  $T : C \rightarrow C$  is uniformly  $\lambda$ -Lipschitzian if for some  $\lambda > 0$ ,  $\|T^n x - T^n y\| \leq \lambda \|x - y\|$  for all  $x, y \in C$  and for all  $n = 1, 2, 3, \dots$ . Moreover, it is asymptotically nonexpansive if for a sequence  $\{k_n\} \subset [0, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = 0$ , we have  $\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|$  for all  $x, y \in C$  and for all  $n = 1, 2, 3, \dots$ . Denote and define by  $F(T) = \{x \in C : Tx = x\}$ , the set of fixed points of  $T$ . If  $F(T) \neq \emptyset$ , then  $T$  is called asymptotically quasi-nonexpansive if for a sequence  $\{k_n\} \subset [0, \infty)$

---

\* Corresponding author.

E-mail addresses: [hfdin@yahoo.com](mailto:hfdin@yahoo.com) (H. Fukhar-ud-din), [safeerhussain5@yahoo.com](mailto:safeerhussain5@yahoo.com) (S.H. Khan).

with  $\lim_{n \rightarrow \infty} k_n = 0$ , we have  $\|T^n x - p\| \leq (1 + k_n)\|x - p\|$  for all  $x \in C$ ,  $p \in F(T)$  and  $n = 1, 2, 3, \dots$ .

A comparison of the above definitions makes it clear that an asymptotically nonexpansive mapping must be uniformly  $\lambda$ -Lipschitzian as well as asymptotically quasi-nonexpansive but the converse does not hold. In recent years, Mann and Ishikawa iterative methods have been studied extensively by many authors to solve one-parameter nonlinear operator equations as well as variational inequalities in Hilbert and Banach spaces. Liu [6] introduced iterative methods with error terms. Later on, Xu [14] improved these methods by giving more satisfactory error terms. Both methods constitute generalizations of Mann and Ishikawa (one mapping) iterative methods. The case of two mappings in iterative processes has also remained under study since Das and Debata [1] gave and studied a two mappings scheme. Also see, for example, [12] and [5]. Note that two mappings case, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see, for example, [11]. Recently the authors [2] studied an iterative process with errors in the sense of Liu [6] for two asymptotically nonexpansive mappings in a uniformly convex Banach space and in [4], we studied an iterative process with errors in the sense of Xu [14] for nonexpansive mappings under weaker conditions in a uniformly convex Banach space. In this manuscript, we take into account a two-step iterative process with errors in the sense of Xu [14] and investigate the approximation of common fixed points of asymptotically quasi-nonexpansive mappings in general Banach spaces and apply it to get convergence results in uniformly convex Banach spaces under the conditions weaker than those found in the contemporary literature. As a matter of fact, we shall study the following process with error terms:

Let  $C$  be a nonempty convex subset of a normed space  $E$  and  $S, T : C \rightarrow C$  be two mappings. The sequence  $\{x_n\}$  is defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T^n x_n + c'_n v_n, \quad n \geq 1, \end{cases} \quad (1.1)$$

with  $\{u_n\}, \{v_n\}$  bounded sequences in  $C$ ,  $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$  sequences in  $[0, 1]$  such that  $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$  for all  $n \geq 1$ . Both Mann and Ishikawa iterative processes with errors as well as without errors can be obtained from this process as special cases by suitably choosing the mappings and the parameters.

In the sequel, we shall need the following lemmas.

**Lemma 1.** [3] *Let  $\{r_n\}, \{s_n\}, \{t_n\}$  be three nonnegative sequences satisfying  $r_{n+1} \leq (1 + s_n)r_n + t_n$  for all  $n \geq 1$ . If  $\sum_{n=1}^{\infty} s_n < \infty$  and  $\sum_{n=1}^{\infty} t_n < \infty$ , then  $\lim_{n \rightarrow \infty} r_n$  exists.*

**Lemma 2.** [9] *Suppose that  $E$  is a uniformly convex Banach space and  $0 < p \leq t_n \leq q < 1$  for all  $n \geq 1$ . Suppose further that  $\{x_n\}$  and  $\{y_n\}$  are sequences of  $E$  such that  $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$ ,  $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$  and  $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$  hold for some  $r \geq 0$ . Then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .*

We shall also make use of the following which we call condition  $(A')$ .

**Condition (A').** Two mappings  $S, T : C \rightarrow C$ , where  $C$  is a subset of a normed space  $E$ , are said to satisfy condition (A') if there exists a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(r) > 0$  for all  $r \in (0, \infty)$  such that either  $\|x - Sx\| \geq f(d(x, F))$  or  $\|x - Tx\| \geq f(d(x, F))$  for all  $x \in C$  where  $d(x, F) = \inf\{\|x - p\| : p \in F = F(S) \cap F(T)\}$ .

Note that the above condition (A') defined by the authors reduces to condition (A) [13] when  $S = T$ . Maiti and Ghosh [7] and Tan and Xu [13] have approximated fixed points of a nonexpansive mapping  $T$  by Ishikawa iterates under the condition (A).

## 2. Convergence in general Banach spaces

In this section we will prove a theorem in a general Banach space which generalizes certain result. This theorem will be applied, in next section, to obtain a result which, in turn, will generalize a number of results. We start with the following lemma.

**Lemma 3.** Let  $C$  be a nonempty closed convex subset of a normed space  $E$  and let  $S, T : C \rightarrow C$  be uniformly  $\lambda$ -Lipschitzian asymptotically quasi-nonexpansive mappings with  $\{k_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} k_n < \infty$ . Define iterative process  $\{x_n\}$  as in (1.1) with  $\{u_n\}, \{v_n\}$  bounded sequences in  $C$  and  $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$  sequences in  $[0, 1]$  satisfying

$$a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n, \quad \sum_{n=1}^{\infty} c_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c'_n < \infty.$$

If  $F = F(S) \cap F(T) \neq \emptyset$ , then  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F$ . Further, denote  $d_n = \|x_n - T^n x_n\|$  and  $d'_n = \|x_n - S^n x_n\|$ . Then

$$\|x_{n+1} - Sx_{n+1}\| \leq d'_{n+1} + \lambda d'_n + \lambda(\lambda + 1)[d'_n + \lambda d_n + (\lambda c'_n + c_n)Q]$$

and

$$\|x_{n+1} - Tx_{n+1}\| \leq d_{n+1} + \lambda d_n + \lambda(\lambda + 1)[d'_n + \lambda d_n + (\lambda c'_n + c_n)Q]$$

for some  $Q > 0$ .

**Proof.** Let  $p \in F$ . Since  $\{u_n\}, \{v_n\}$  are bounded, therefore there exists  $M > 0$  such that  $\max\{\sup_{n \geq 1} \|u_n - p\|, \sup_{n \geq 1} \|v_n - p\|\} \leq M$ .

Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|a_n x_n + b_n S^n y_n + c_n u_n - p\| \\ &\leq a_n \|x_n - p\| + b_n \|S^n y_n - p\| + c_n \|u_n - p\| \\ &\leq a_n \|x_n - p\| + b_n (1 + k_n) \|y_n - p\| + c_n M \\ &\leq [a_n + a'_n b_n (1 + k_n) + b_n b'_n (1 + k_n)^2] \|x_n - p\| + (c_n + b_n c'_n) M \\ &\leq (1 + k_n)^2 \|x_n - p\| + (c_n + b_n c'_n) M. \end{aligned}$$

Thus by Lemma 1,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F$  and hence  $\{x_n\}$  is bounded. Choose  $Q > 0$  such that  $\max\{\sup_{n \geq 1} \|x_n - u_n\|, \sup_{n \geq 1} \|x_n - v_n\|\} \leq Q$ . Now consider

$$\begin{aligned}
\|x_n - x_{n+1}\| &= \|(a_n + b_n + c_n)x_n - (a_n x_n + b_n S^n y_n + c_n u_n)\| \\
&= \|b_n(x_n - S^n y_n) + c_n(x_n - u_n)\| \\
&\leq \|x_n - S^n x_n\| + \|S^n x_n - S^n y_n\| + c_n \|x_n - u_n\| \\
&\leq d'_n + \lambda \|x_n - y_n\| + c_n Q \\
&= d'_n + \lambda \|b'_n(x_n - T^n x_n) + c'_n(x_n - v_n)\| + c_n Q \\
&\leq d'_n + \lambda b'_n d_n + (\lambda c'_n + c_n) Q \\
&\leq d'_n + \lambda d_n + (\lambda c'_n + c_n) Q.
\end{aligned}$$

That is,

$$\|x_n - x_{n+1}\| \leq d'_n + \lambda d_n + (\lambda c'_n + c_n) Q. \quad (2.1)$$

Next,

$$\begin{aligned}
\|x_{n+1} - Sx_{n+1}\| &\leq \|x_{n+1} - S^{n+1}x_{n+1}\| + \|Sx_{n+1} - S^{n+1}x_{n+1}\| \\
&\leq d'_{n+1} + \lambda \|x_{n+1} - S^n x_{n+1}\| \\
&= d'_{n+1} + \lambda \|(x_{n+1} - x_n) + (x_n - S^n x_n) + (S^n x_n - S^n x_{n+1})\| \\
&\leq d'_{n+1} + \lambda [\|x_{n+1} - x_n\| + \|x_n - S^n x_n\| + \lambda \|x_n - x_{n+1}\|] \\
&= d'_{n+1} + \lambda d'_n + \lambda(\lambda + 1) \|x_{n+1} - x_n\|.
\end{aligned}$$

That is,

$$\|x_{n+1} - Sx_{n+1}\| \leq d'_{n+1} + \lambda d'_n + \lambda(\lambda + 1) \|x_{n+1} - x_n\|. \quad (2.2)$$

Substituting (2.1) into (2.2), we obtain that

$$\|x_{n+1} - Sx_{n+1}\| \leq d'_{n+1} + \lambda d'_n + \lambda(\lambda + 1) [d'_n + \lambda d_n + (\lambda c'_n + c_n) Q]. \quad (2.3)$$

Similarly we can prove that

$$\|x_{n+1} - Tx_{n+1}\| \leq d_{n+1} + \lambda d_n + \lambda(\lambda + 1) [d'_n + \lambda d_n + (\lambda c'_n + c_n) Q]. \quad (2.4)$$

This completes the proof.  $\square$

**Lemma 4.** Let  $C$  be a nonempty closed subset of a normed space  $E$  and let  $T : C \rightarrow C$  be an asymptotically quasi-nonexpansive mapping. Then  $F(T)$ , the set of fixed points of  $T$ , is closed.

**Proof.** The proof is a routine work.  $\square$

**Lemma 5.** Let  $C$  be a nonempty closed convex subset of a normed space  $E$  and let  $S, T : C \rightarrow C$  be asymptotically quasi-nonexpansive mappings with sequence  $\{k_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} k_n < \infty$ . Define an iterative process  $\{x_n\}$  as given in (1.1) with  $\{u_n\}, \{v_n\}$  bounded sequences in  $C$  and  $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$  sequences in  $[0, 1]$  satisfying

$$a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n, \quad \sum_{n=1}^{\infty} c_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c'_n < \infty.$$

If  $F = F(S) \cap F(T) \neq \emptyset$ , then

$$\|x_{n+m} - p\| \leq R \left( \|x_n - p\| + \sum_{j=n}^{\infty} h_j \right) \quad (2.5)$$

for all  $m, n \geq 1$ , for all  $p \in F$  and for some  $R > 0$  and  $h_j = (c_j + b_j c'_j)M$ .

**Proof.** As calculated in Lemma 3, we have

$$\|x_{n+1} - p\| \leq (1 + k_n)^2 \|x_n - p\| + h_n \quad \text{where } h_n = (c_n + b_n c'_n)M.$$

It is well known that  $1 + x \leq e^x$  for all  $x \geq 0$ . Using it for the above inequality, we get

$$\begin{aligned} \|x_{n+m} - p\| &\leq (1 + k_{n+m-1})^2 \|x_{n+m-1} - p\| + h_{n+m-1} \\ &\leq e^{2k_{n+m-1}} \|x_{n+m-1} - p\| + h_{n+m-1} \\ &\leq e^{2k_{n+m-1}} [(1 + k_{n+m-2})^2 \|x_{n+m-2} - p\| + h_{n+m-2}] + h_{n+m-1} \\ &\leq e^{2(k_{n+m-1} + k_{n+m-2})} \|x_{n+m-2} - p\| + e^{2k_{n+m-1}} [h_{n+m-2} + h_{n+m-1}] \\ &\vdots \\ &\leq e^{2 \sum_{j=n}^{n+m-1} k_j} \|x_n - p\| + e^{2 \sum_{j=n}^{n+m-1} k_j} \sum_{j=n}^{n+m-1} h_j \\ &\leq R \left[ \|x_n - p\| + \sum_{j=n}^{n+m-1} h_j \right], \quad \text{where } R = e^{2 \sum_{j=n}^{n+m-1} k_j}. \end{aligned}$$

That is,

$$\|x_{n+m} - p\| \leq R \left[ \|x_n - p\| + \sum_{j=1}^{\infty} h_j \right]$$

for all  $m, n \geq 1$ , for all  $p \in F$  and for  $R = e^{2 \sum_{j=n}^{n+m-1} k_j} > 0$  and  $h_j = (c_j + b_j c'_j)M$ .  $\square$

Now we give our main theorem of this paper.

**Theorem 1.** Let  $E$  be a real Banach space and let  $C, S, T, F, \{x_n\}, \{u_n\}, \{v_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$  be as given in Lemma 5. Then  $\{x_n\}$  converges to a point in  $F$  if and only if  $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$  where  $d(x, F) = \inf\{\|x - p\| : p \in F\}$ .

**Proof.** Necessity is obvious. We only prove the sufficiency. Suppose that  $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ . As proved in Lemma 3, we have

$$\|x_{n+1} - p\| \leq (1 + k_n)^2 \|x_n - p\| + (c_n + b_n c'_n)M.$$

This gives

$$d(x_{n+1}, F) \leq [1 + (2 + k_n)k_n]d(x_n, F) + (c_n + b_n c'_n)M.$$

As  $\sum_{n=1}^{\infty} k_n < \infty$  and  $\sum_{n=1}^{\infty} (c_n + b_n c'_n) < \infty$ , therefore  $\lim_{n \rightarrow \infty} d(x_n, F)$  exists by Lemma 1. But by hypothesis  $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ , therefore we must have  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ .

Next we show that  $\{x_n\}$  is a Cauchy sequence. Let  $\epsilon > 0$ . Since  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$  and  $\sum_{n=1}^{\infty} h_n < \infty$ , therefore there exists a constant  $n_0$  such that for all  $n \geq n_0$ , we have

$$d(x_n, F) < \frac{\epsilon}{4R} \quad \text{and} \quad \sum_{j=n_0}^{\infty} h_j < \frac{\epsilon}{6R}.$$

In particular,  $d(x_{n_0}, F) < \frac{\epsilon}{4R}$ . That is,  $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\epsilon}{4R}$ . There must exist  $p^* \in F$  such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{3R}.$$

Now for  $n \geq n_0$ , we have from inequality (2.5) of Lemma 5 that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2R \left[ \|x_{n_0} - p^*\| + \sum_{j=n_0}^{n_0+m-1} h_j \right] \\ &< 2R \left( \frac{\epsilon}{3R} + \frac{\epsilon}{6R} \right) = \epsilon. \end{aligned}$$

Hence  $\{x_n\}$  is a Cauchy sequence in a closed subset  $C$  of a Banach space  $E$ , therefore it must converge to a point in  $C$ . Let  $\lim_{n \rightarrow \infty} x_n = q$ . Now  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$  gives that  $d(q, F) = 0$ .  $F$  is closed as indicated in Lemma 4, therefore  $q \in F$ , and the proof of the theorem is complete.  $\square$

**Remark.** The above theorem generalizes Theorem 1 of Qihou [8] to the case of two asymptotically quasi-nonexpansive mappings.

### 3. Applications

In this section, we apply Theorem 1 proved in the previous section to obtain the strong convergence of the scheme (1.1) under condition (A') which is weaker than the compactness of the domain of the mappings. Actually we prove the following theorem.

**Theorem 2.** Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Let  $S, T : C \rightarrow C$  be uniformly  $\lambda$ -Lipschitzian mappings satisfying condition (A') and

$$\|S^n x - p\| \leq (1 + k_n)\|x - p\|, \quad \|T^n x - p\| \leq (1 + k_n)\|x - p\|$$

for all  $n = 1, 2, 3, \dots$ , where  $\{k_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} k_n < \infty$ . Construct an iterative process  $\{x_n\}$  as in Lemma 5 with  $0 < 1 - \delta \leq b_n$ ,  $b'_n \leq \delta < 1$  for all  $n = 1, 2, 3, \dots$ . If  $F = F(S) \cap F(T) \neq \emptyset$ , then  $\{x_n\}$  converges strongly to a common fixed point of  $S$  and  $T$ .

**Proof.** As proved in Lemma 3,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F$ . Denote this limit by  $c$ . If  $c = 0$ , then the proof of the theorem is over. Suppose  $c > 0$  and  $Q$  is the real number as appeared in Lemma 3.

Now consider

$$\begin{aligned} \|y_n - p\| &\leq \|b'_n(T^n x_n - p + c'_n(v_n - x_n)) + (1 - b'_n)(x_n - p + c'_n(v_n - x_n))\| \\ &\leq b'_n(1 + k_n)\|x_n - p\| + (1 - b'_n)\|x_n - p\| + c'_n\|v_n - x_n\| \\ &\leq \|x_n - p\| + (1 - \delta)k_n\|x_n - p\| + c'_n(1 - \delta)Q. \end{aligned}$$

Taking  $\limsup$  on both the sides, we have

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq c. \quad (3.1)$$

Note that  $\|S^n y_n - p + c_n(u_n - x_n)\| \leq (1 + k_n)\|y_n - p\| + c_n Q$  gives with the help of (3.1) that

$$\limsup_{n \rightarrow \infty} \|S^n y_n - p + c_n(u_n - x_n)\| \leq c. \quad (3.2)$$

Also  $\|x_n - p + c_n(u_n - x_n)\| \leq \|x_n - p\| + c_n Q$  gives that

$$\limsup_{n \rightarrow \infty} \|x_n - p + c_n(u_n - x_n)\| \leq c. \quad (3.3)$$

Further,  $\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c$  can be written as

$$\lim_{n \rightarrow \infty} \|b_n(S^n y_n - p + c_n(u_n - x_n)) + (1 - b_n)(x_n - p + c_n(u_n - x_n))\| = c. \quad (3.4)$$

Hence by Lemma 2 together with (3.2)–(3.4), we obtain that

$$\lim_{n \rightarrow \infty} \|x_n - S^n y_n\| = 0. \quad (3.5)$$

Thus

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\leq \|x_n - S^n y_n\| + (1 + k_n)\|y_n - p\| \end{aligned}$$

gives that

$$\lim_{n \rightarrow \infty} \|y_n - p\| = c. \quad (3.6)$$

That is,

$$\lim_{n \rightarrow \infty} \|b'_n(T^n x_n - p + c'_n(v_n - x_n)) + (1 - b'_n)(x_n - p + c'_n(v_n - x_n))\| = c.$$

Since

$$\limsup_{n \rightarrow \infty} \|T^n x_n - p + c'_n(v_n - x_n)\| \leq c \quad (3.7)$$

and

$$\limsup_{n \rightarrow \infty} \|x_n - p + c'_n(v_n - x_n)\| \leq c, \quad (3.8)$$

so again by Lemma 2, together with (3.6)–(3.8), we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0. \quad (3.9)$$

Further, observe that

$$\begin{aligned} \|x_n - S^n x_n\| &\leq \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ &\leq \lambda \|x_n - y_n\| + \|S^n y_n - x_n\| \\ &= \lambda \|b'_n(x_n - T^n x_n) + c'_n(x_n - v_n)\| + \|S^n y_n - x_n\| \\ &\leq (1 - \delta)\lambda \|x_n - T^n x_n\| + c'_n \lambda Q + \|S^n y_n - x_n\|. \end{aligned}$$

By (3.5) and (3.9), we achieve that

$$\lim_{n \rightarrow \infty} \|x_n - S^n x_n\| = 0. \quad (3.10)$$

Now Lemma 3 applied upon (3.9) and (3.10) yields

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0 = \lim_{n \rightarrow \infty} \|x_n - Tx_n\|. \quad (3.11)$$

From condition (A') and (3.11), we get

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$$

or

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

In both the cases,

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0.$$

Since  $f: [0, \infty) \rightarrow [0, \infty)$  is a nondecreasing function satisfying  $f(0) = 0$ ,  $f(r) > 0$  for all  $r \in (0, \infty)$ , therefore we have

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

Now all the conditions of Theorem 1 are satisfied, therefore by its conclusion  $\{x_n\}$  converges strongly to a point of  $F$ .  $\square$

As every asymptotically nonexpansive mapping is uniformly  $\lambda$ -Lipschitzian asymptotically quasi-nonexpansive, therefore not only Theorem 2 of Khan and Takahashi [5] but also its following extension are the immediate consequences of the above theorem. Note that condition of compactness in Theorem 2 of Khan and Takahashi [5] has also been relaxed to condition (A').

**Corollary 1.** *Let  $C$  be a nonempty bounded closed convex subset of a uniformly convex Banach space  $E$ . Let  $S, T: C \rightarrow C$  satisfy condition (A') and*

$$\|S^n x - S^n y\| \leq (1 + k_n)\|x - y\|, \quad \|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|$$

*for all  $n = 1, 2, 3, \dots$ , where  $\{k_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} k_n < \infty$ . Construct an iterative process  $\{x_n\}$  as in Lemma 5 with  $0 < 1 - \delta \leq b_n$ ,  $b'_n \leq \delta < 1$  for all  $n = 1, 2, 3, \dots$ . If  $F = F(S) \cap F(T) \neq \emptyset$ , then  $\{x_n\}$  converges strongly to a common fixed point of  $S$  and  $T$ .*

**Remark.** If we take  $S = T$  in Theorem 2, then this theorem reduces to Ishikawa type convergence theorem for quasi-asymptotically nonexpansive mappings and if we put  $c'_n = 0$  and  $T = I$  in the same theorem, then we obtain Mann type convergence. Similarly, Corollary 1 reduces to Ishikawa type and Mann type convergence theorems for asymptotically nonexpansive mappings. Thus Theorem 3 of Qihou [8], Theorem 2 of Khan and Fukhar [4], Theorem 2 of Senter and Dotson [10], Theorem 1 of Maiti and Ghosh [7] and Theorem 2 of Schu [9] are all special cases of Theorem 2 above.

## References

- [1] G. Das, J.P. Debata, Fixed points of quasi-nonexpansive mappings, Indian J. Pure Appl. Math. 17 (1986) 1263–1269.
- [2] H. Fukhar-ud-din, S.H. Khan, Convergence of two-step iterative scheme with errors for two asymptotically nonexpansive mappings, Int. J. Math. Math. Sci. 37 (2004) 1965–1971.
- [3] D.I. Igbokwe, Approximation of fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces, J. Inequal. Pure Appl. Math. 3 (1) (2002), Article 3; online: [http://jipam.vu.edu.au/v3n1/043\\_01.html](http://jipam.vu.edu.au/v3n1/043_01.html).



- [4] S.H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, *Nonlinear Anal.* 61 (8) (2005) 1295–1301.
- [5] S.H. Khan, W. Takahashi, Approximating common fixed points of two asymptotically nonexpansive mappings, *Sci. Math. Jpn.* 53 (1) (2001) 143–148.
- [6] L.S. Liu, Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.* 194 (1) (1995) 114–125.
- [7] M. Maiti, M.K. Ghosh, Approximating fixed points by Ishikawa iterates, *Bull. Austral. Math. Soc.* 40 (1989) 113–117.
- [8] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mappings with errors member, *J. Math. Anal. Appl.* 259 (2001) 18–24.
- [9] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.* 43 (1991) 153–159.
- [10] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* 44 (2) (1974) 375–380.
- [11] W. Takahashi, Iterative methods for approximation of fixed points and their applications, *J. Oper. Res. Soc. Japan* 43 (1) (2000) 87–108.
- [12] W. Takahashi, T. Tamura, Convergence theorems for a pair of nonexpansive mappings, *J. Convex Anal.* 5 (1) (1998) 45–58.
- [13] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* 178 (1993) 301–308.
- [14] Y. Xu, Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations, *J. Math. Anal. Appl.* 224 (1998) 91–101.